Eur. Phys. J. Special Topics **222**, 895–903 (2013)

DOI: 10.1140/epjst/e2013-01892-1

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Regular Article

Dynamic transitions in Domany-Kinzel cellular automata on small-world network

A.V. Mahajan[1,a], M. Ali Saif[2,b], and P.M. Gade[3,c]

[1] Centre for Modeling and Simulation, University of Pune, Pune 411007, India
[2] PG Department of Physics, University of Amran, Yemen
[3] PG Department of Physics, Rashtrasant Tukdoji Maharaj Nagpur University, Nagpur 440033, India

Received 22 March 2013 / Received in final form 3 May 2013
Published online 11 July 2013

Abstract. We study Domany-Kinzel cellular automata on small-world network. Every link on a one dimensional chain is rewired and coupled with any node with probability p. We observe that, the introduction of long-range interactions does not remove the critical character of the model and the system still exhibits a well-defined phase transition to absorbing state. In case of directed percolation (DP), we observe a very anomalous behavior as a function of size. The system shows long lived metastable states and a jump in order parameter. This jump vanishes in thermodynamic limit and we recover second-order transition. The critical exponents are not equal to the mean-field values even for large p. However, for compact directed percolation(CDP), the critical exponents reach their mean-field values even for small p.

1 Introduction

Modern theory of critical phenomena is a major success story in statistical physics. It has improved our understanding of phase transitions in equilibrium systems by leaps and bounds. The concepts such as scalings, renormalization group and universality classes have enriched the comprehension and added insight in these systems [1]. Recently, considerable efforts have been devoted to understanding phase transitions in nonequilibrium systems as well. One of the challenging problem is to classify dynamical phase transitions of such systems into different universality classes. The concept of universality is one of the most useful and important concepts in study of phase transitions. It allows us to group different systems to small number of classes and allows us to know the essential and not essential details of the systems. Directed percolation (DP) class [2] is perhaps the most well-studied transition in nonequilibrium phase transitions. This transition is observed in a vast number of systems, for example Domany-kinzel (DK) automata [3], Contact process [4], Ziff-Gulari-Barshad

[a] e-mail: mahajan@cms.unipune.ac.in
[b] e-mail: masali73@gmail.com
[c] e-mail: prashant.m.gade@gmail.com

model for heterogeneous catalysis [5], stochastic prisoner's dilemma [6] and Pair Contact Process [7]. The list is so long that now it is generically believed that all the continuous phase transition from fluctuating phase to single absorbing state are in universality class of directed percolation (DP) [2].

Of course, the phase transitions are usually studied on a d dimensional lattice. Of late, various other topologies have also been studied in this context. This includes complex networks such as small-world and scale- free networks [8].

In this work, we study the DK automata on small-world network [9]. For completeness, we give the definition of the model. It is defined as follows: one starts with one dimensional lattice. Each site has k nearest-neighbors on either side. This gives $2k$ connections. We disconnect the links of any site to its nearest neighbors with probability p and connected to randomly chosen lattice site. This model is proposed to mimic real life situations in which nonlocal connections exist along with predominantly local connections. Of course, in such situations, there is a possibility that this network disorder may not be completely averaged when we average over several configurations. Thus there is a theoretical justification for studying annealed model as well. However, in this work we focus on dynamics with quenched connections.

Several equilibrium transitions and few non-equilibrium transitions have been studied on small-world networks. The behavior is markedly different in equilibrium and non-equilibrium systems. In equilibrium systems such as Ising model and XY model, it is seen that the critical behavior of Ising and XY system is same as that of mean-field case even for infinitesimally small values of p [10,11]. Even percolation model which can be mapped on Potts model in the limit $q \to 1$, shows same critical behavior as mean-field model for infinitesimal values of p [12,13].

This strongly suggests that the behavior for any $p \neq 0$ is the same as mean-field system for these models. Why do we expect mean-field like behavior for small-world lattice. The reason is that in these systems, characteristic distances between nodes grow as logarithm of system size. (In a d-dimensional cartesian network with N sites ($N = L^d$), characteristic length scale $L = N^{1/d}$ and since logarithm is slower than power-law, it is effectively infinite dimensional network.) In equilibrium statistical physics, the system shows mean-field behavior in high dimensions. Basically, we expect that the fluctuations are not important they get averaged out very quickly in presence of nonlocal connections. Hence, such behavior is expected for non-equilibrium phase transitions as well.

Nonequilibrium systems have been much less studied from this viewpoint. This transition has been investigated in majority-vote model [14], Ising transition for directed small-world network [15] and transition to phase synchronization in a model for the spread of infection [16] These studies suggest that, these systems behave as mean-field only at finite value of p. For a very slowly driven system, we may observe a dynamic transition for infinitesimal p as in equilibrium systems [17]. There are some other studies on networks such as transition to clustered state in coupled map lattice with small-world connections [18]. In this case it has been claimed that persistence acts as an order parameter. However, very non-universal exponents are obtained and they do not observe any monotonic dependence of the exponent on p.

However, even for $p = 1$, when the network is locally tree-like and purely random, it is not clear is system reaches its mean-field values. The $p = 1$ limit of this system is similar to Erdős−Rényi model. (The difference is that the exact number of neighbors is $2k$ for small-world model with $p = 1$ whereas it is a mean number of neighbors for Erdős−Rényi model.) It has been shown by Melo and coworkers that the universality class of 3-state majority-vote model is significantly different from mean-field model for small k and it approaches mean-field values only when number of neighbors is same as lattice size [19]. On the other hand, Campos et al. have obtained a steady approach of critical exponents to mean-field values with increasing p in a 2-state majority vote model [14]. There are also reports that transition becomes first-order for large values of p [15]. Thus there are four possible scenarios *a*) The critical exponents reach their mean-field values even for infinitesimal p. *b*) The critical exponents monotonically change as a function of p and reach their mean-field values for $p = 1$. *c*) Critical exponents do not reach mean-field values even for $p = 1$. *d*) The transition becomes first-order for large p.

In this work, we study Domany-Kinzel automata on small-world network. This is a nonequilibrium system and displays a well known DP phase transition. However, for $p_2 = 1$ (to be defined later), this model gets mapped on CDP (compact directed percolation) which is a equilibrium system. Studying this system on small-world network gives us opportunity to study the behavior of both equilibrium and non-equilibrium systems under the influence of added nonlocal connections, as one varies p in the region of phase space corresponding to DP class. When the model corresponds to Compact Directed Percolation (CDP), the system displays transition in mean-field class even at small p as observed in equilibrium systems. (We must mention that for $p_2 = 1$, the mapping to an equilibrium process is not at all clear for small-world system.) Situation is more complicated for $p_2 \neq 1$.

For DP, there is extremely anomalous behavior for even for large N, (even of order 10^4). There are extremely long-lived metastable states. *However, careful analysis shows that the parameter regime over which the metastable states appear shrinks and the gap in order parameter disappears with N.* Thus we believe that transition remains second-order. We find that exponents are not same as their mean-field values for finite p. They do not even approach these values for $p = 1$. In fact, they do not change much as a function of p. Thus universality class of small-world network is distinct from mean-field class. In fact, the exponent seems independent of value of p_2 and p, provided $p_2 \neq 1$.

2 Model

As mentioned above, Domany-Kinzel (DK) Automata is one of the earliest model displaying directed percolation (DP) [3]. It is defined on a diagonal square lattice and evolves by parallel updates in original model. The lattice sites are either empty ($s = 0$) (dead) or occupied by a particle ($s = 1$) (active site). At each time step, the state of each sites $s_i(t+1)$ depends on the state of its two nearest neighbors in the previous time step according to certain conditional transition probability $P[s_i(t+1)|s_{i-1}(t), s_{i+1}(t)]$. These probabilities depend on two parameters and are defined by

$$\begin{aligned} P[1|0,0] &= 0, \\ P[1|1,0] = P[1|0,1] &= p_1, \\ P[1|1,1] &= p_2, \end{aligned} \tag{1}$$

where $P[0|.,.] = 1 - P[1|.,.]$. Notice that the DK model depends on two percolation probabilities p_1 and p_2. Again, a state $s_i(T) = 0$ for all i at some time T will lead to $s_i(t) = 0$ for $t \geq T$. Thus all sites being zero is an absorbing state. Since an absorbing state is an object of interest, density of active sites is an appropriate order parameter of this system.

We consider DK automaton on small-world network, when rewiring probability i.e. $p = 0$, each site i is connected to sites $i+1$ and $i-1$. Assume periodic boundary conditions. For $p \neq 0$, we rewire these 2 connections with probability p and connect site i to a randomly chosen sites on the lattice. Unlike original model, we update all sites at the same time. As expected, nature of transition does not change for

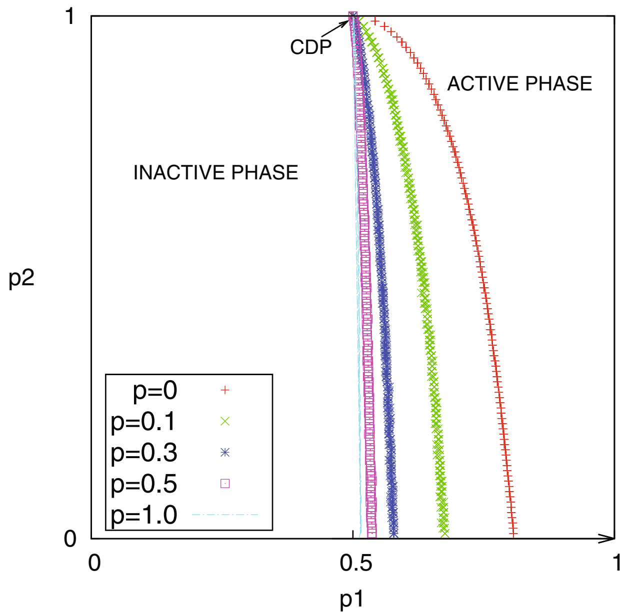

Fig. 1. Schematic phase diagram of DK automaton at different values of p. The area in the left of each curve is an inactive phase while the other side is active phase. The consecutive lines from right to left are for $p = 0.0, 0.1, 0.3, 0.5$ and $p = 1$.

$p = 0$ and it clearly displays DP exponents in 1-d. It is expected that in presence of nonlocal connections, the universality class of phase transition will change. We make a detailed study the phase transition from active phase to inactive phase for small-world lattice.

The case $p_2 = 1$ is a special case. For $p_2 = 1$, there is no possibility of creation of 0 in a connected domain of 1's. This universality class is known as compact directed percolation (CDP). In 1-d, this is equivalent to ferromagnetic Glauber-Ising model at zero temperature [20]. As mentioned before, this equivalence is not clear for small-world lattice. The connections are not even symmetric. As pointed out by Sanchez, this makes an important difference [15].

3 Phase diagram

We simulate DK for different values of p in the full plane (p_1, p_2). In the Fig. 1, we plot the phase space of DK automaton for various values of p. For $p = 0$, system does not exactly reduce to original DK model since we do not update only odd lattice sites followed by only even lattice sites. The critical line is very close to that for original DK model and the critical exponents are as expected for DP transition in 1-d. The critical behavior along the whole phase transition line (except for its upper terminal point) is that of DP for $p = 0$. The critical line which separates the active phase from inactive phase shifts continuously with p. For $p \neq 0$, there is a continuous change in the critical point with p and it approaches $p_1 = 1/2$. For $p_2 = 1$, the critical point remains the same at $p_1 = 1/2$ irrespective of the value of p. For $p_2 < 1$, the critical line tends to $p_1 = 1/2$ as $p \to 1$. As mentioned above, $p_2 = 1$ is a special case since there are two possible absorbing state and one observes a jump from one absorbing state to other at $p_1 = 1/2$.

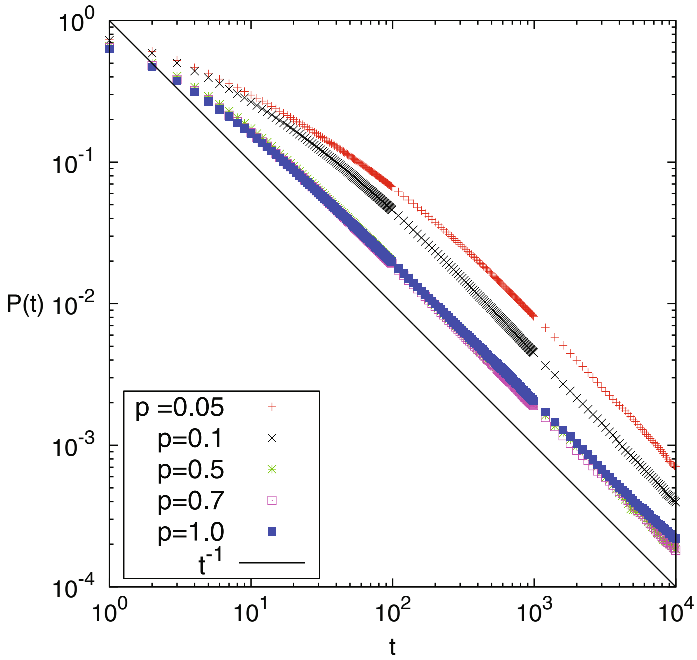

Fig. 2. Survival probability $P(t)$ is plotted as a function of t for various values of p for CDP. We find $1/t$ behavior in all the cases. The lattice size $L = 10^5$. We average over at least 10^5 configurations.

4 Results

The nature of phase transition on the line separating the active and inactive phases could be first-order or second-order. For $p = 0$, we know that transition is continuous. This nature is retained for small values of p. However, even finding nature and order of phase transition is not easy for higher values of p. For p = 0.1, the transition is continuous and we have found several critical points. The fraction of active sites at the critical point is expected to go as $\rho(t) \sim t^{-\alpha}$ at the critical point.

First we discuss CDP. For CDP, the critical point remains same at all values of p. It is $p_1 = 0.5, p_2 = 1$. At this point, the survival probability $P(t)$ that a cluster grown from a single seed survives till time t is the order parameter. The exponent quantifying decay of this probability is extremely useful for studying CDP.

The survival probability $P(t)$ decays as $t^{-\delta}$ with $\delta = 1/2$ in 1-d and $\delta = 1$ in the mean-field limit. We find that this probability decays as $1/t$ for all values of p for large enough lattice size. We have plotted survival probability as a function of time for $p = 0.05, 0.1, 0.5, 0.7, 1$ in Fig. 2. We observe that the all of them have $1/t$ dependence. Thus behavior of CDP on small-world is similar to the one observed in equilibrium systems. We must mention that simulations for very large lattice size are necessary to obtain accurate exponents.

For DP, the dynamical behavior is more complex for larger values of p. For continuous phase transitions, we expect to find a power-law decay of order parameter at critical point. On the other hand, for first-order transitions, asymptotic value of order parameter displays sharp and discontinuous jump at the critical point in thermodynamic limit. Since we need to inspect the "asymptotic" values in "thermodynamic" limit, it is clear that unless we can have some systematic way in which the $N \to \infty$ and $t \to \infty$ limits can be inferred, we cannot have firm conclusions. Often finite-size

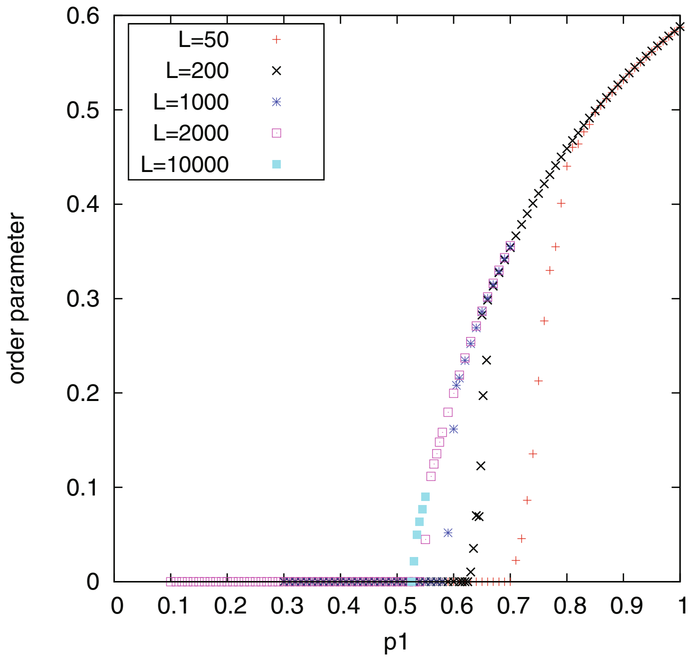

Fig. 3. Asymptotic value of order parameter vs p_1 for different Lattice sizes.

scaling helps us to approach this limit. For finite lattice sizes, there is a systematic departure from asymptotic curve and we can scale it to get appropriate exponents for second-order transitions. For larger values of p, we need to investigate carefully to deduce the nature of transition. At the outset, the nature of transition is not very clear and it seems like first-order transition for a small lattice size. The first-order transition is characterized by various features. There is a gap in the order parameter at the critical point, there are long lived metastable states and there is hysteresis. Besides, binder cumulant shows a minimum near critical point. However, for absorbing phase transitions, binder cumulant is undefined in the absorbing phase. Similarly hysteresis studies cannot be carried out since the system will get stuck in the absorbed phase in those parameter values. At times, authors have tried to induce a small source of fluctuations in the system so that hysteresis and Binder cumulant studies can be carried out. Alternatively, they add a small constant to the value of order parameter. Since first possibility perturbs the system the second changes numerical values, we do not like either option and we have refrained from carrying out such studies. The waiting time after which system settles into absorbing state clearly diverges near critical point.

If we study the behavior of order parameter as a function of time for various lattice sizes, the order parameter hits a plateau and does not change over a long period of time which is a characteristic of long lived metastable states in first-order transitions. However, this is followed by a power-law decay and not exponential decay of order parameter.

We have made detailed studies for $p = 0.5, p_2 = 0.3$. The lifetime of metastable state diverges near the critical point. However, the range over which the divergence occurs reduces with increasing N. Thus the error due to presence of metastable states decreases for larger N.

In Fig. 3, asymptotic value of order parameter is plotted for various lattice sizes. We have waited for at least 10^6 timesteps and for longer times 1.2×10^7 for parameter

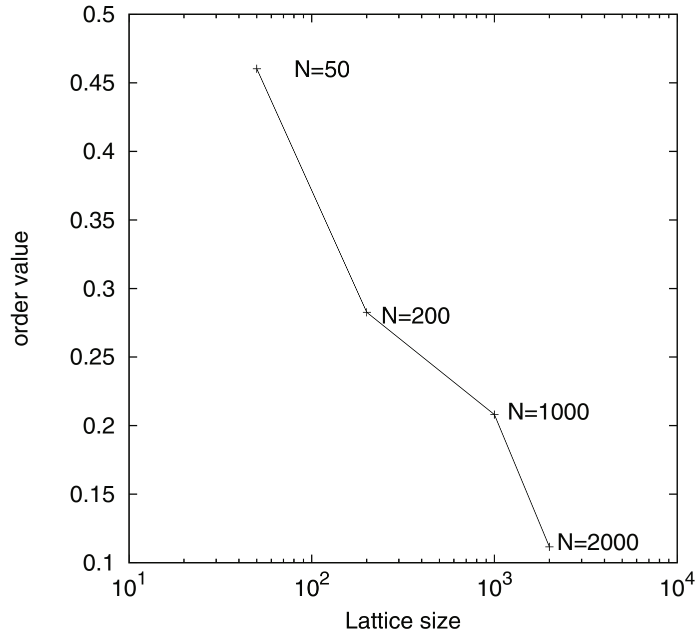

Fig. 4. Jump in order parameter vs. Lattice size.

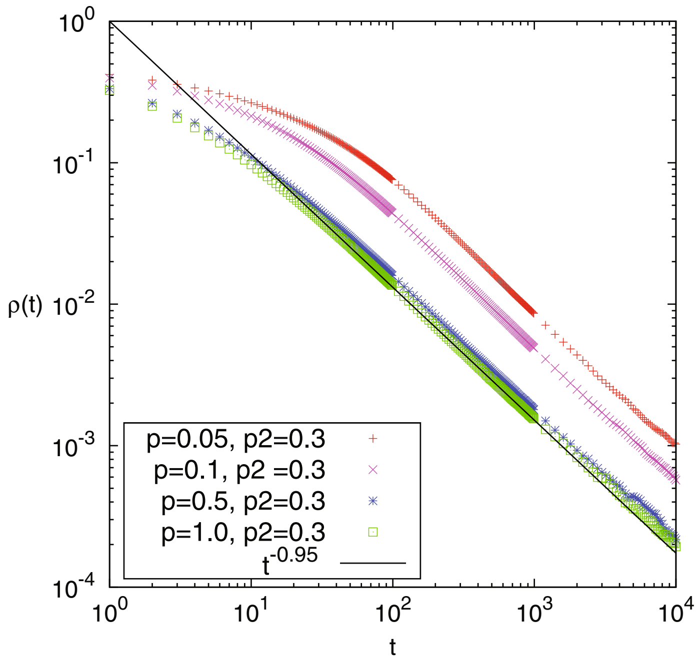

Fig. 5. Fraction of active sites $\rho(t)$ as a function of time for different values of p at critical point.

values close to critical point. We have plotted the value of jump in order parameter $\Delta\rho$ for various lattice sizes in Fig. 4. We observe that though there is a jump near critical point, *the jump reduces with increasing lattice size.* Absence of a finite jump in limit $N \to \infty$ confirms that the transition is not first-order. Since the system has

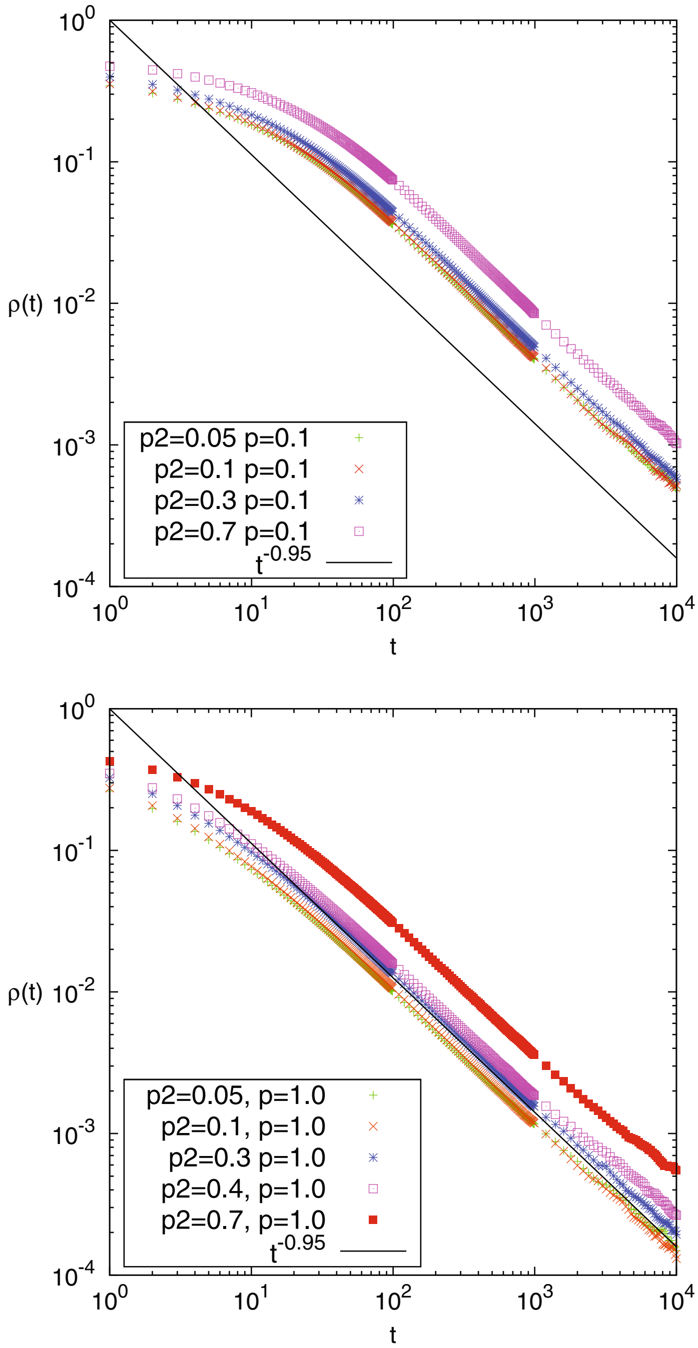

Fig. 6. Fraction of active sites $\rho(t)$ as a function of time for $p = 0.1$ and $p = 1.0$ for different values of p_2 at critical point.

a very anomalous size-dependence, it is not possible to carry out finite-size scaling. We assume that the 10^7 is a large enough size to estimate the asymptotic value of exponents (which is a very good assumption for most of the systems). We find that the exponent α for decay of $\rho(t)$ at the critical point is far from its one-dimensional value even for $p = 0.05$. The value of the exponent is roughly in the range $0.94 - 0.95$. With increasing p, the exponent does not increase any further and even for $p = 1$, we observe same value of exponent. In Fig. 5, we have plotted $\rho(t)$ as a function of t at critical point for various values of p keeping $p_2 = 0.3$. We can see that the graphs are parallel to each other and the exponent is practically unchanged. We have carried out simulations for $p = 0.1$ and $p = 1.0$ for different values of p_2. we observe that the exponent does not change much either with p or p_2 and remains at $0.94 - 0.95$ range throughout Fig. 6. This value is significantly different from the expected mean-field exponent of 1.0. Thus the exponent approaches a random-network exponent for small p and does not change with increasing p. However, for CDP, the exponent is 1 even for small values of p.

5 Conclusion

We conclude that DK automata on small-world lattice does not approach mean-field universality class for the case of DP, even for $p = 1$, when underlying topology is a random network. However, for CDP the results are consistent with previous observations for equilibrium systems and this system approaches mean-field universality for any nonzero value of p. Thus the possible behaviors of dynamic phase transitions are far richer and vary significantly from system to system for nonequilibrium phase transitions. One could argue that adiabatic drive, annealed connections and increasing the number of connections could help to average out the fluctuations faster and system could display a transition in mean-field universality class. Our preliminary investigations indicate that annealed connections or updating odd and even sites at different times does not change the critical exponents and system does not approach mean-field class. Thus this universality class is fairly robust.

We thank DST for financial assistance.

References

1. H.E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, USA, 1987)
2. H. Hinrichsen, Adv. Phys. **49**, 815 (2000) and references therein
3. E. Domany, W. Kinzel, Phys. Rev. Lett. **53**, 311 (1984)
4. T.M. Liggett, *Interacting Particle Systems* (Springer, New York, 1985)
5. R.M. Ziff, E. Gulari, Y. Barshad, Phys. Rev. Lett. **56**, 2553 (1986)
6. M. Ali. Saif, P. M. Gade, J. Stat. Mech: Th. Ex., P03016 (2010)
7. I. Jensen, Phys. Rev. Lett. **70**, 1465 (1993)
8. See e.g. A. Barrat, M. Barthélemy, A. Vespignani, *Dynamical Processes on Complex Networks* (Cambridge University Press, 2008)
9. D.J. Watts, S.H. Strogatz, Nature (London) **393**, 440 (1998)
10. H. Hong, B.J. Kim, M.Y. Choi, Phys. Rev. E **66**, 018101 (2002)
11. K. Medvedyeva, P. Holme, P. Minnhagen, B.J. Kim, Phys. Rev. E **67**, 036118 (2003)
12. C. Moore, M.E.J. Newman, Phys. Rev. E **61**, 5678 (2000)
13. C. Moore, M.E.J. Newman, Phys. Rev. E **62**, 7059 (2000)
14. P.R.A. Campos, V.M. de Oliveira, Phys. Rev. E **67**, 026104 (2003)
15. A.D. Sánchez, J.M. López, M.A. Rodríguez, Phys. Rev. Lett. **88**, 048701 (2002)
16. M. Kuperman, G. Abramson, Phys. Rev. Lett. **86**, 2909 (2001)
17. P.M. Gade, S. Sinha, Phys. Rev. E **72**, 052903 (2005)
18. A.V. Mahajan, P.M. Gade, Phys. Rev. E **81**, 056211 (2010)
19. D.F.F. Melo, L.F.C. Pereira, F.G.B. Moreira, J. Stat. Mech: Th. Ex., P11032 (2010)
20. N. Menyhárd, G. Ódor, J. Phys. Math. **31**, 6771 (1998)