Introduction
to
Fourier transform

Fourier series

- Decompose *f(t)* in terms of sine and cosine *bases* function
- Similarity with *i*, *j* and *k* unit vectors. Unique representation in 3-D
- Euclidean space: $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}.$
- Orthonormal sets: $\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = 1$ and $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0,$
- Orthonormality condition: $A_x = \hat{i}.\vec{A},\ A_y = \hat{j}.\vec{A}$ and $A_z = \hat{k}.\vec{A}.$

Fourier series

- Function *f(t)* is periodic

$$L\ (f(t+L) = f(t))$$

- Finite energy

$$\int_{t_0}^{t_0+L} |f(t)|^2 dt < \infty,$$

- Fourier series

$$f(t) = \frac{a_o}{2} + \sum_{r=1}^{\infty}\left[a_r \cos\left(\frac{2\pi r t}{L}\right) + b_r \sin\left(\frac{2\pi r t}{L}\right)\right].$$

Fourier series

Orthogonal basis sets:

$$\int_{t_o}^{t_o+L} \sin\left(\frac{2\pi p t}{L}\right) \cos\left(\frac{2\pi r t}{L}\right) dt = 0 \quad \text{for all } p \text{ and } r,$$

$$\int_{t_o}^{t_o+L} \cos\left(\frac{2\pi p t}{L}\right) \cos\left(\frac{2\pi r t}{L}\right) dt = \begin{cases} L & \text{for } p = r = 0, \\ \frac{1}{2}L & \text{for } p = r > 0, \\ 0 & \text{for } p \neq r, \end{cases}$$

$$\int_{t_o}^{t_o+L} \sin\left(\frac{2\pi p t}{L}\right) \sin\left(\frac{2\pi r t}{L}\right) dt = \begin{cases} 0 & \text{for } p = r = 0, \\ \frac{1}{2}L & \text{for } p = r > 0, \\ 0 & \text{for } p \neq r, \end{cases}$$

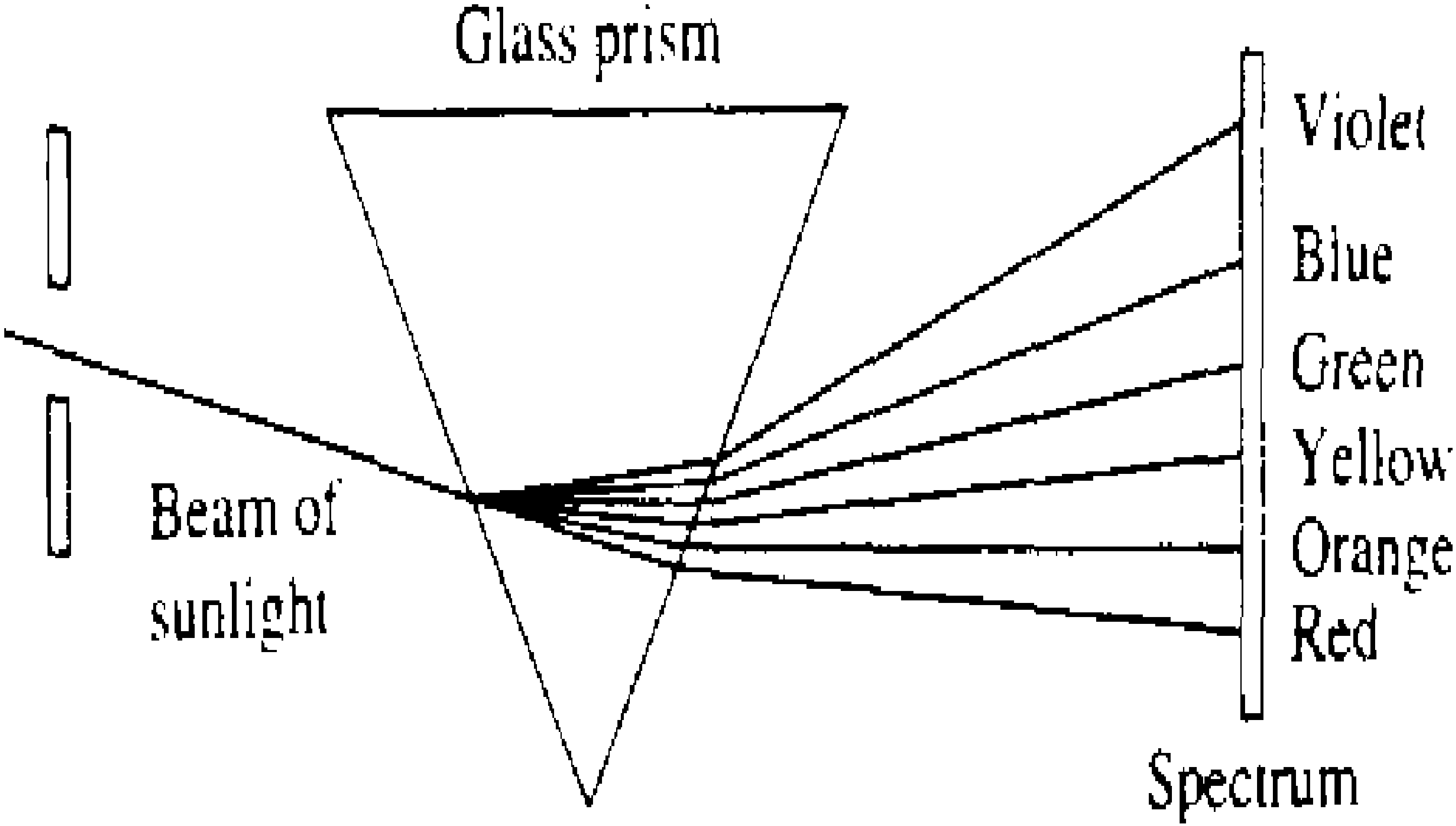
Extraction of coefficients:

$$a_r = \frac{2}{L}\int_{t_o}^{t_o+L} f(t)\cos\left(\frac{2\pi r t}{L}\right) dt \ ,$$

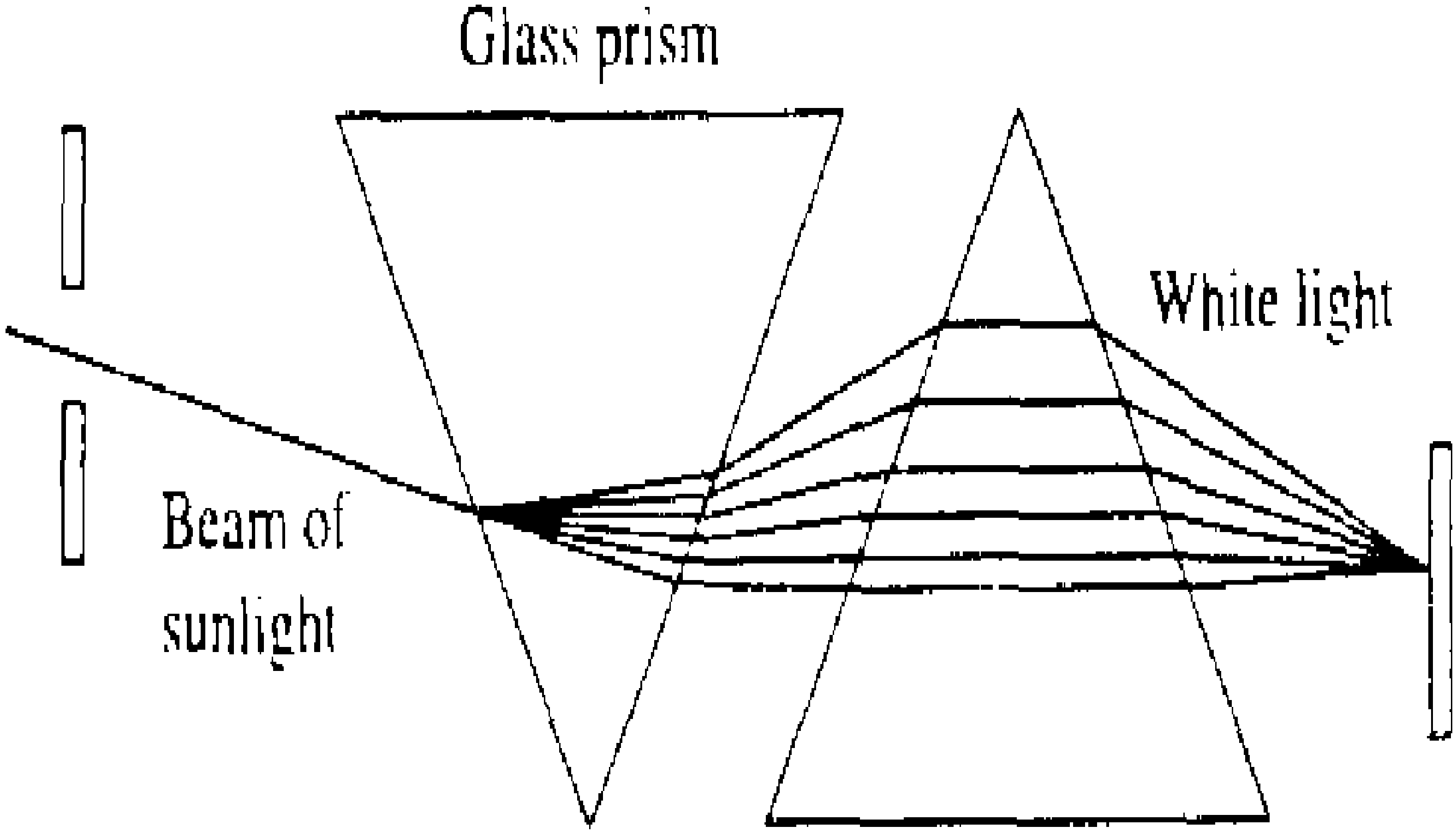
$$b_r = \frac{2}{L}\int_{t_o}^{t_o+L} f(t)\sin\left(\frac{2\pi r t}{L}\right) dt \ .$$

Fourier series: Interpretation

A *mathematical* prism



Fourier series: Interpretation



(b)

Fourier transforms (FT)

f(t): not periodic, but decreases at infinity

Forward FT:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

Inverse FT:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega$$

General notation

Fourier transforms: specific notation

Direct (Forward) transform:

$$X(F) = \int \; x(t)\, e^{(-j2\pi Ft)}\, dt$$

Inverse transform:

$$x(t) = \int X(F)\, e^{(j2\pi Ft)}\, dF$$

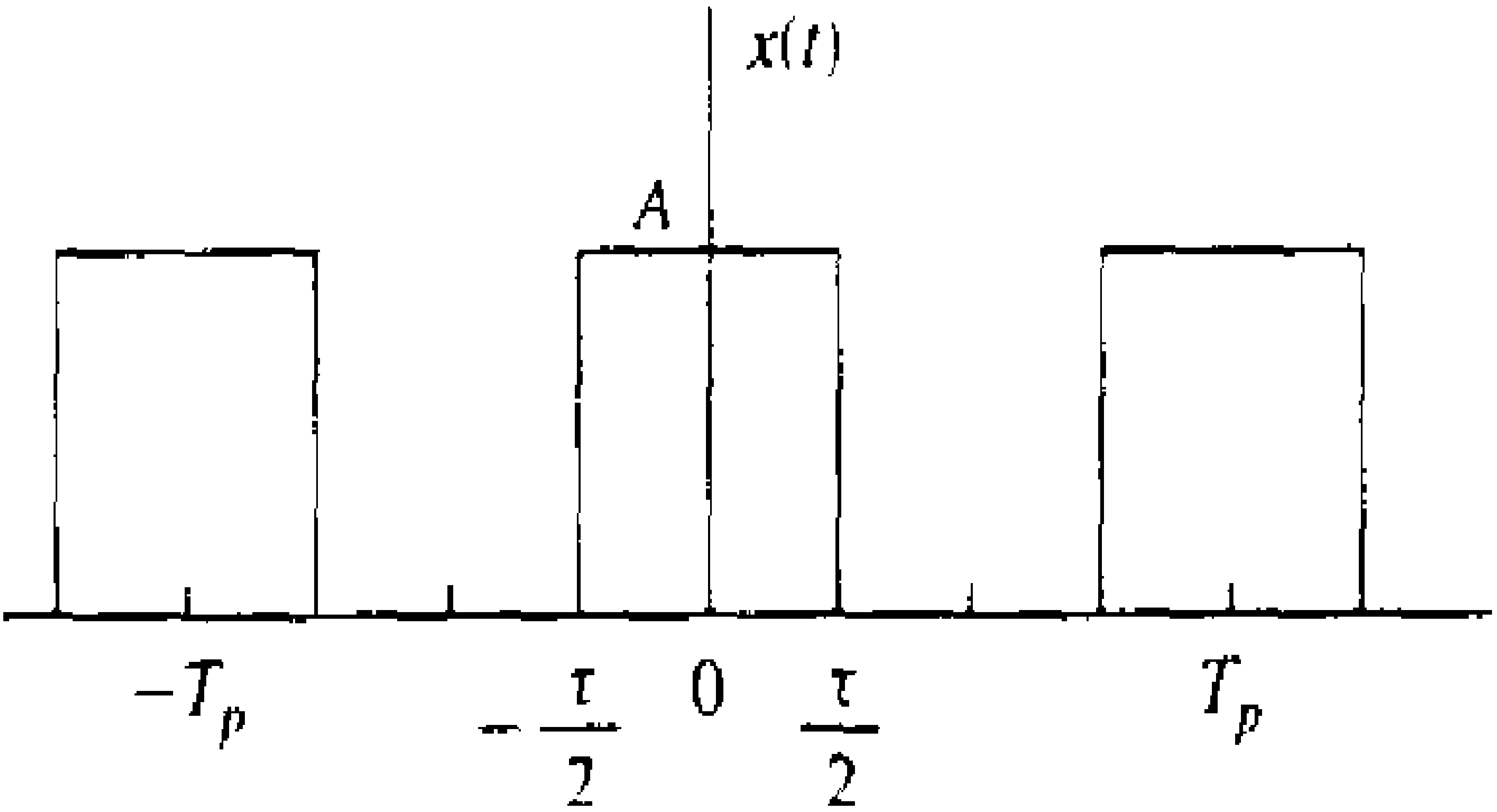
Fourier transforms: Examples

Fourier transform of rectangular train pulse defined as

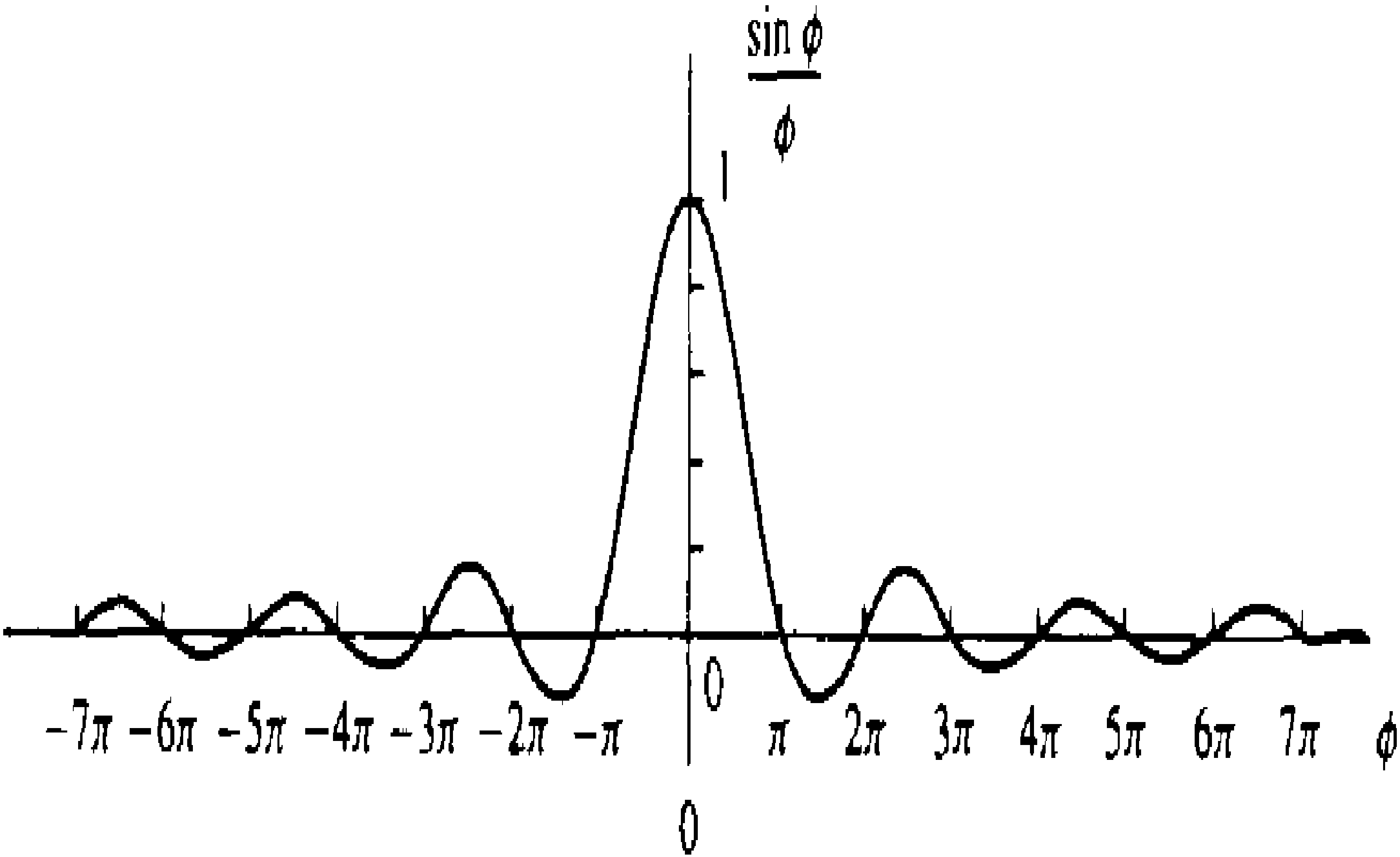
$$x(t) = \{ A, |t| \leq \tau/2$$
$$|t| \geq \tau/2\}$$

Fourier transforms: Solution

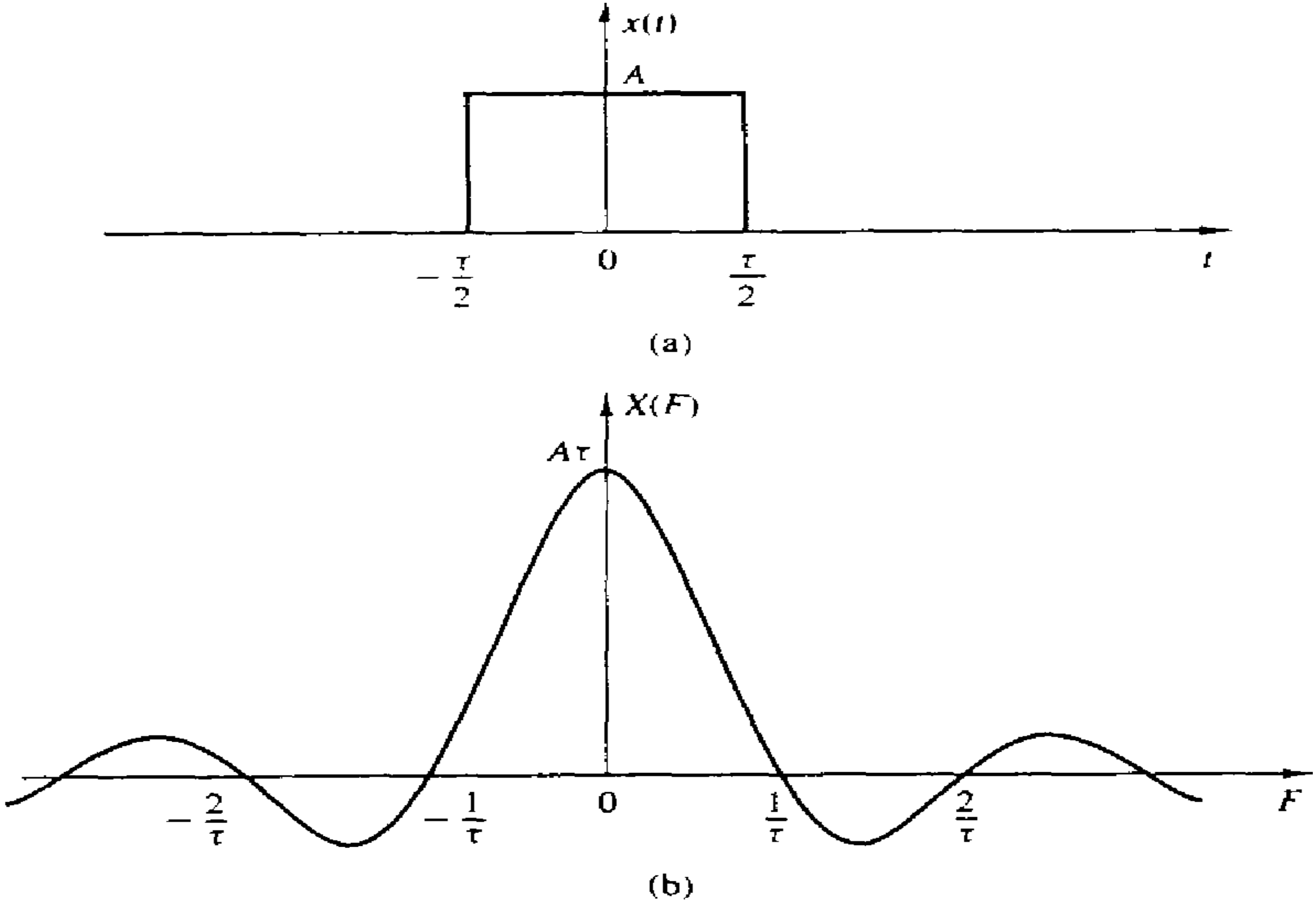
$$X(F) = \int_{-\tau/2}^{\tau/2} A\, e^{(-j2\pi Ft)}dt = A\tau\,\{\sin(\pi F\tau)/\,(\pi F\tau)\}$$

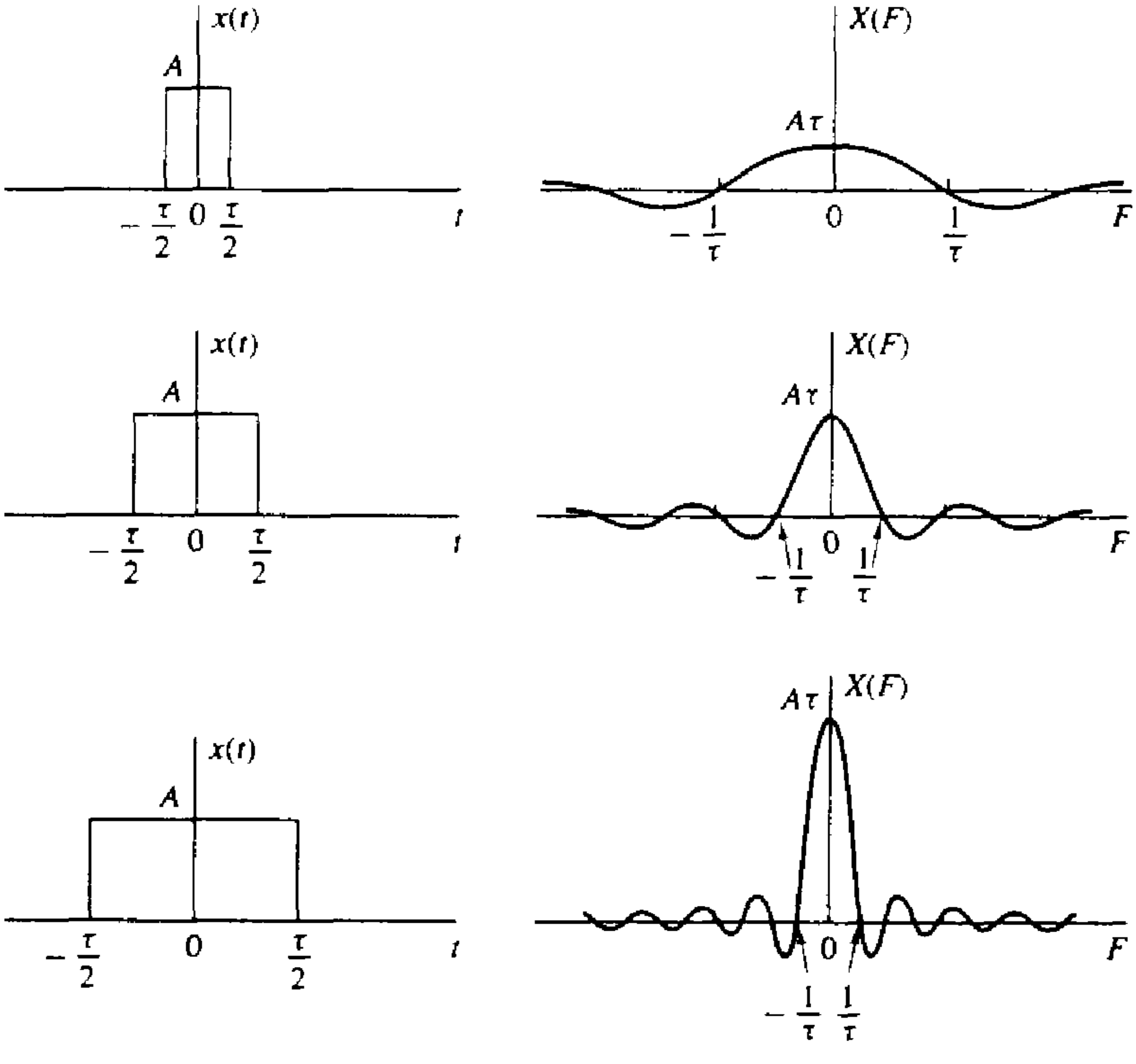


x(t)
A
−T_p
−τ/2
0
τ/2
T_p



$\frac{\sin\phi}{\phi}$
1
0
-7π -6π -5π -4π -3π -2π $-\pi$ π 2π 3π 4π 5π 6π 7π ϕ
0


x(t)
A
$-\frac{\tau}{2}$
0
$\frac{\tau}{2}$
t
(a)
X(F)
Aτ
$-\frac{2}{\tau}$
$-\frac{1}{\tau}$
0
$\frac{1}{\tau}$
$\frac{2}{\tau}$
F
(b)


x(t)
A
$-\frac{\tau}{2}$ 0 $\frac{\tau}{2}$
t
X(F)
Aτ
$-\frac{1}{\tau}$ 0 $\frac{1}{\tau}$
F
x(t)
A
$-\frac{\tau}{2}$ 0 $\frac{\tau}{2}$
t
X(F)
Aτ
$-\frac{1}{\tau}$ 0 $\frac{1}{\tau}$
F
x(t)
A
$-\frac{\tau}{2}$ 0 $\frac{\tau}{2}$
t
X(F)
Aτ
$-\frac{1}{\tau}$ 0 $\frac{1}{\tau}$
F

Power spectrum: Parseval's theorem

$$\int_{-\infty}^{+\infty} |X(F)|^2 \, dF = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

Total Power is conserved.

Power spectrum: Parseval's theorem

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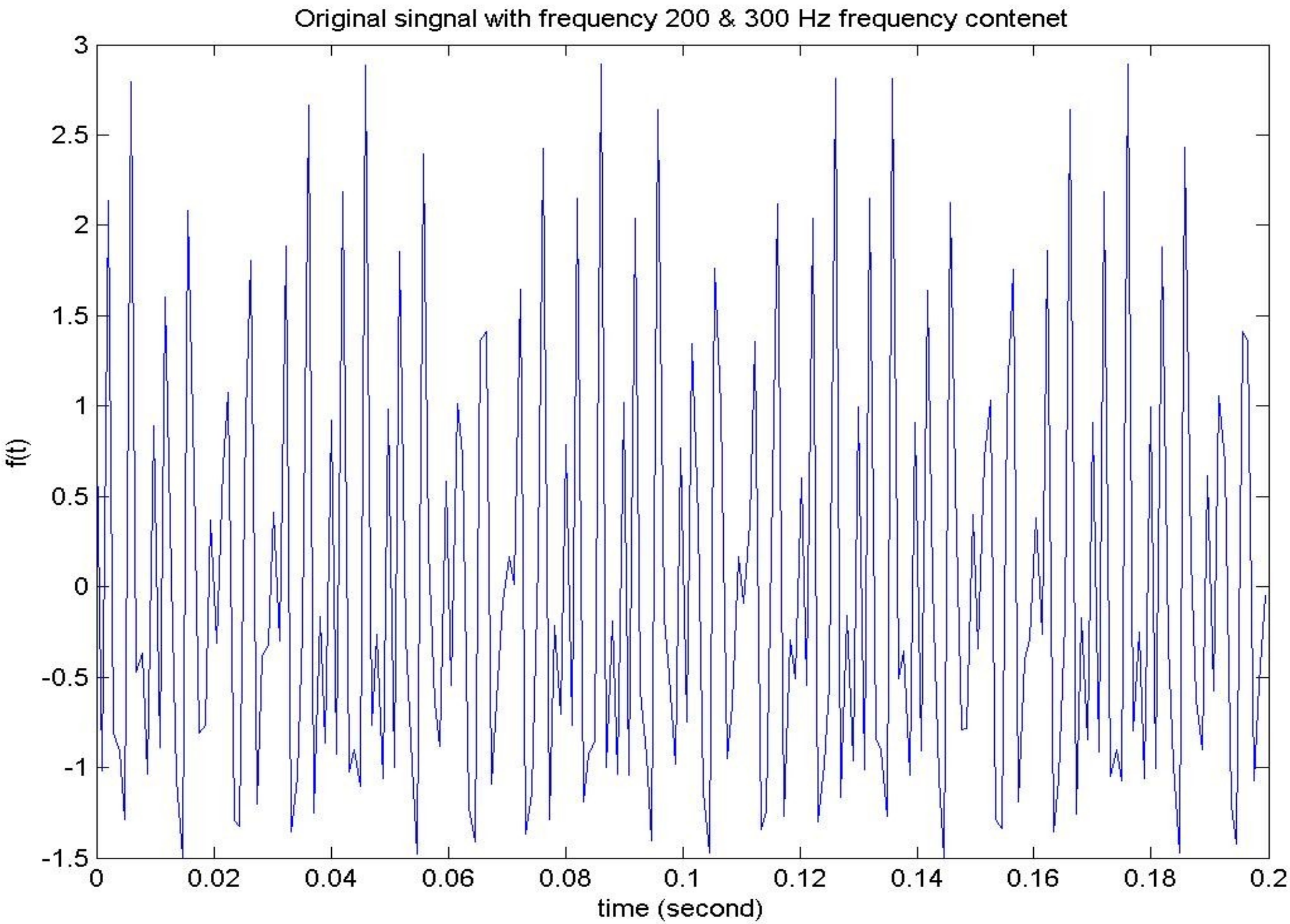
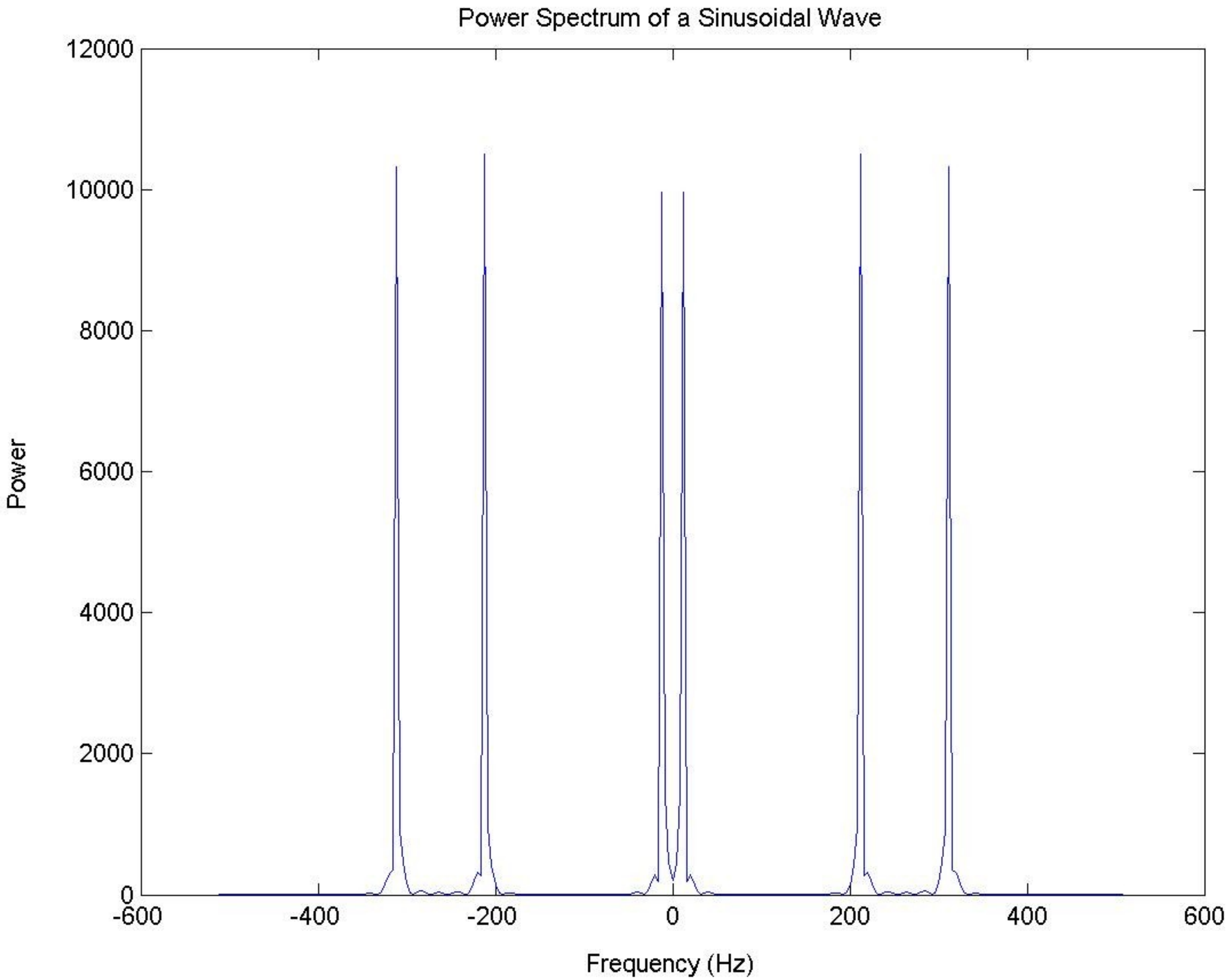
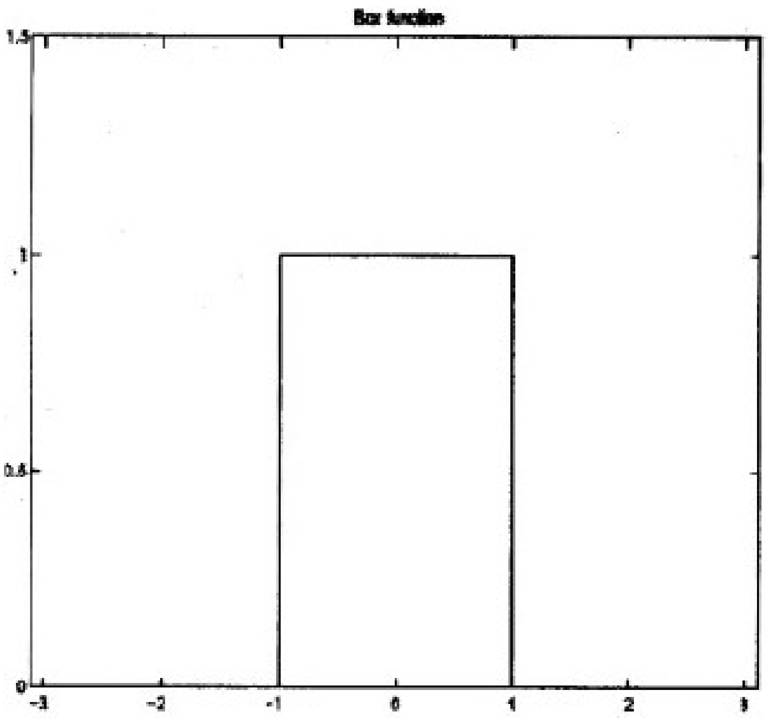
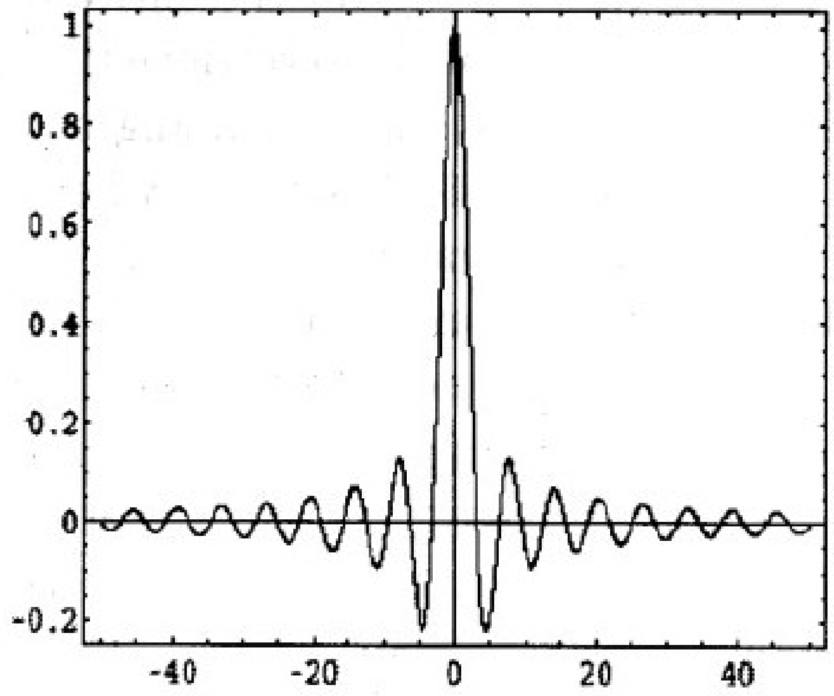

Original singnal with frequency 200 & 300 Hz frequency contenet
f(t)
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
time (second)


Power Spectrum of a Sinusoidal Wave
Power
Frequency (Hz)
12000
10000
8000
6000
4000
2000
0
-600
-400
-200
0
200
400
600

Fourier transforms: disadvantages

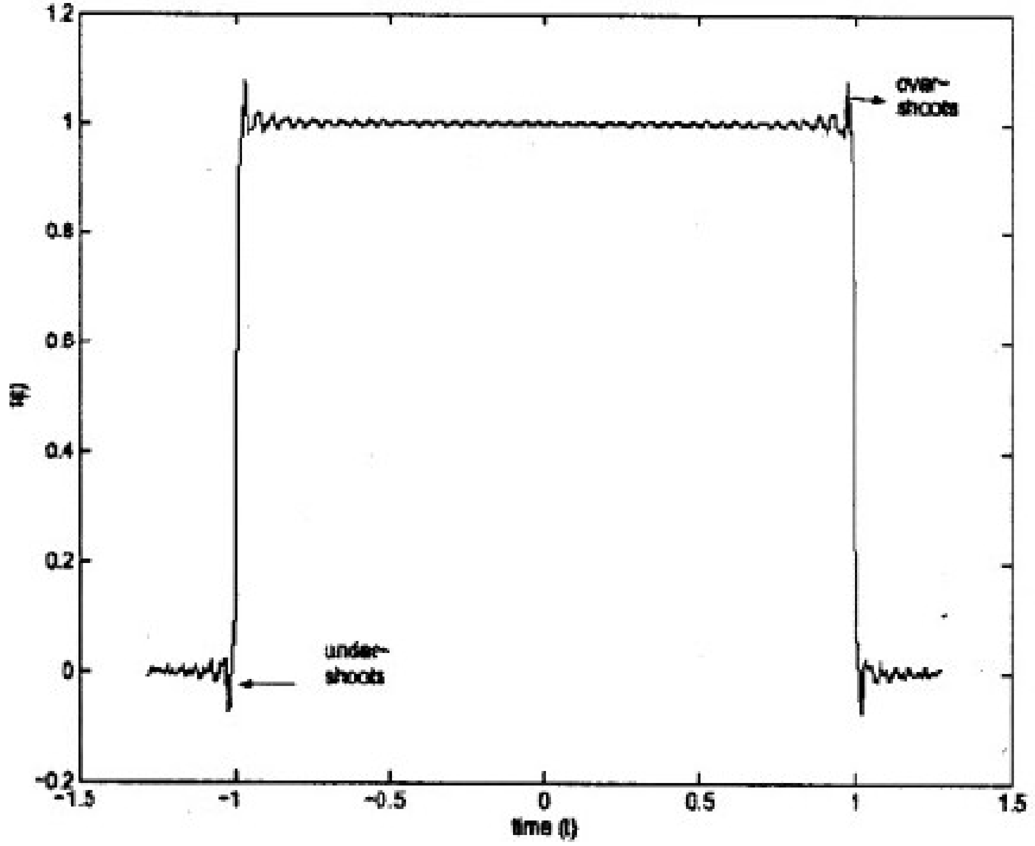




Fourier transforms: disadvantages

What happens after removal of handful of Fourier coefficients?

Gibb’s phenomenon



Fourier transforms: disadvantages

Conclusions:

- Not suitable for *transient* signals with sharp changes.
- Sine and cosine basis sets: *de-localized*
- *Time* information difficult to retrieve.
- Solution: Use a thorn to remove a thorn!!
- Use *Wavelet* transforms

Discrete Fourier transformation (DFT)

- Discrete Time domain
- Complexity: $O(N^2)$

- Danielson and Lanczos Leema (1942)

FFT: Radix-2 Algorithm

Forward transform:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad 0 \leq k \leq N-1$$

$$W_N = e^{-j2\pi/N}$$

Inverse transform

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-nk} \qquad 0 \le n \le N-1$$

Symmetry property

$$W_N^{k+N/2} = -W_N^k$$

Periodicity property

$$W_N^{k+N} = W_N^k$$

Divide and conquer rule

$$X(k) = \sum_{m=0}^{(N/2)-1} f_1(m) W_{N/2}^{km} + W_N^k \sum_{m=0}^{(N/2)-1} f_2(m) W_{N/2}^{km}$$

$$= F_1(k) + W_N^k F_2(k) \qquad k = 0, 1, \ldots, N-1$$

Periodicity property

$$X(k) = F_1(k) + W_N^k F_2(k) \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

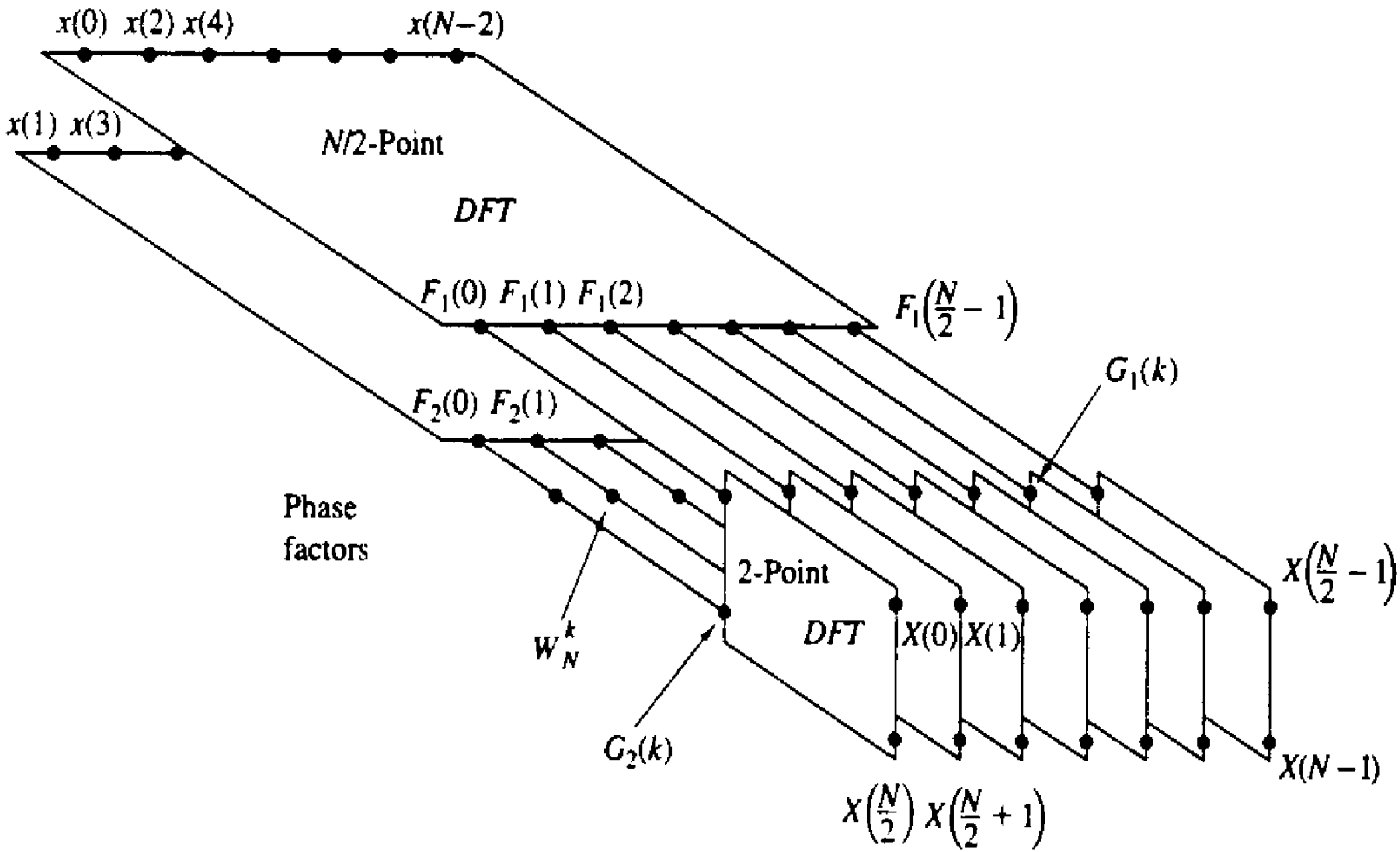
$$X\left(k + \frac{N}{2}\right) = F_1(k) - W_N^k F_2(k) \qquad k = 0, 1, \ldots, \frac{N}{2} - 1$$

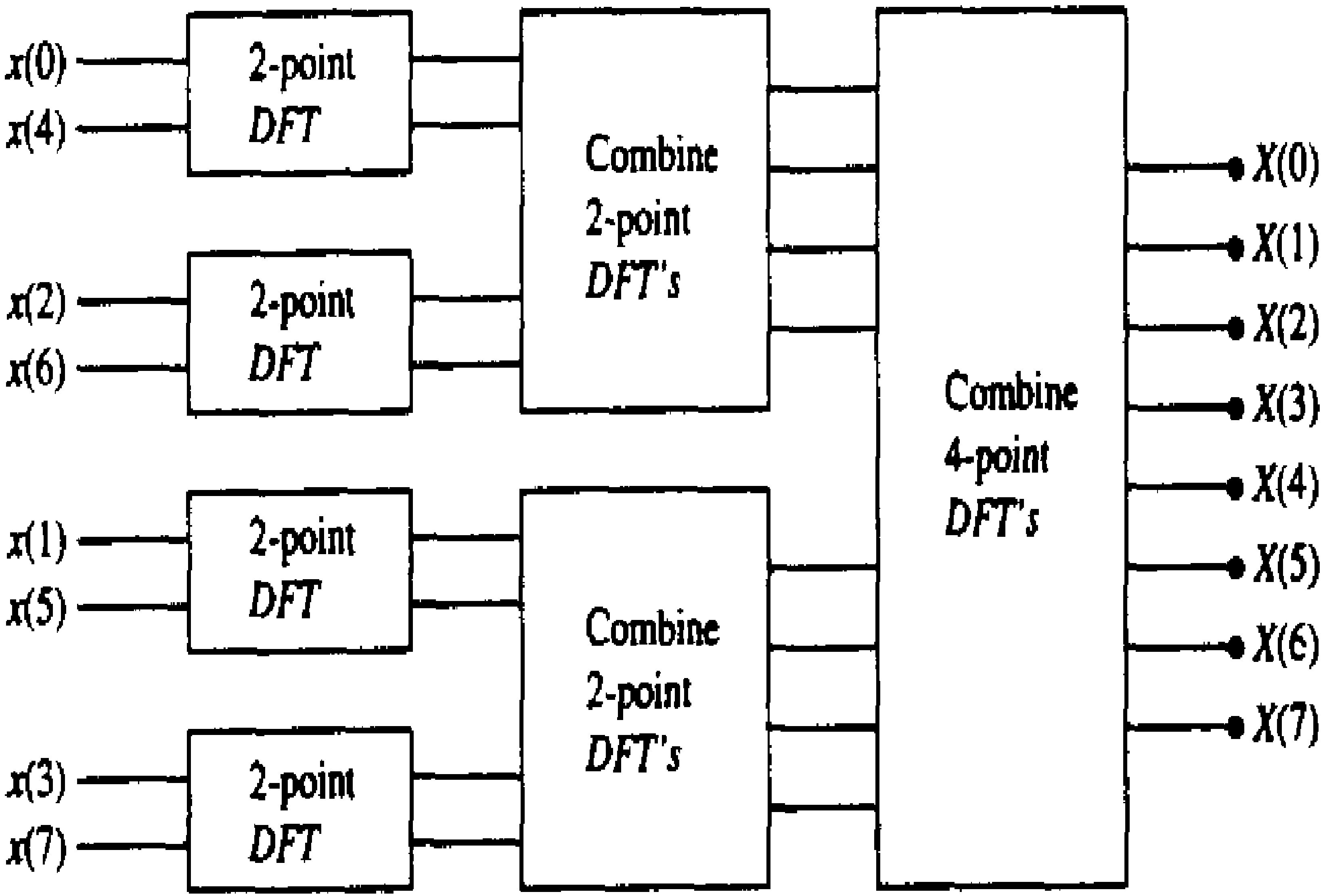
$F_1(k)$ & $F_2(k)$ = $(N/2)^2$ complex multiplications each

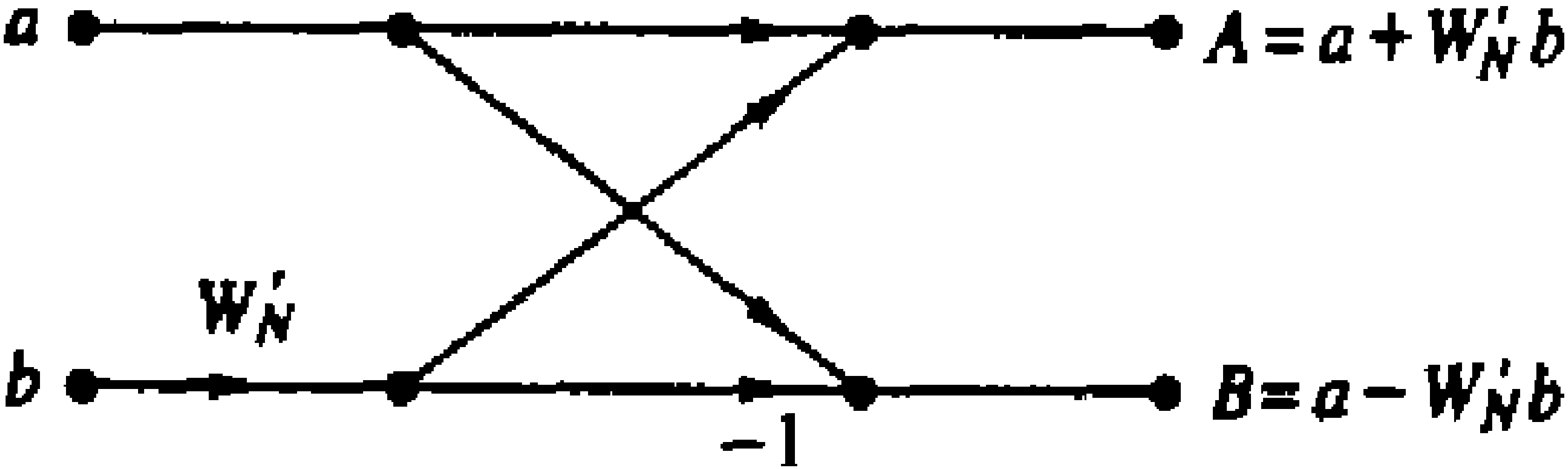
W^k_N = (N/2) complex multiplications

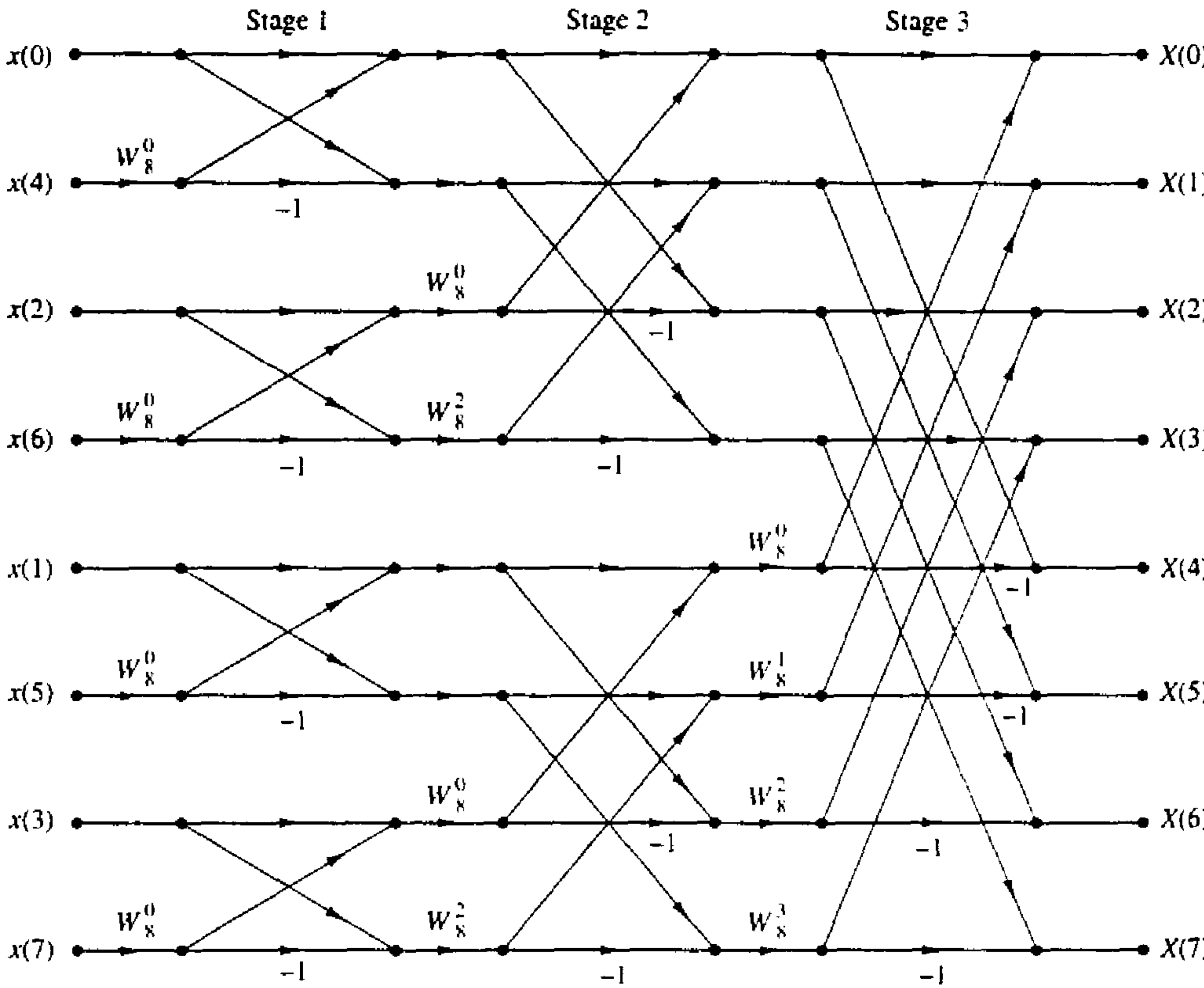
Total = $N^2/2 + N/2$

Recursively apply the algorithm


x(0) x(2) x(4)
x(N−2)
x(1) x(3)
N/2-Point
DFT
F1(0) F1(1) F1(2)
F1(N/2 − 1)
G1(k)
F2(0) F2(1)
Phase
factors
2-Point
DFT
X(0) X(1)
X(N/2 − 1)
W_N^k
G2(k)
X(N − 1)
X(N/2) X(N/2 + 1)


x(0)
x(4)
2-point DFT
x(2)
x(6)
2-point DFT
x(1)
x(5)
2-point DFT
x(3)
x(7)
2-point DFT
Combine 2-point DFT's
Combine 2-point DFT's
Combine 4-point DFT's
X(0)
X(1)
X(2)
X(3)
X(4)
X(5)
X(6)
X(7)


a
b
W_N^r
-1
$A = a + W_N^r b$
$B = a - W_N^r b$


Stage 1
Stage 2
Stage 3
x(0)
x(4)
x(2)
x(6)
x(1)
x(5)
x(3)
x(7)
X(0)
X(1)
X(2)
X(3)
X(4)
X(5)
X(6)
X(7)
W_8^0
W_8^2
W_8^1
W_8^3
−1

References

1) *Digital Signal Processing: Principles, Algorithms and Applications* by John G. Proakis and Dimitris G. Manolakis, Prentice-Hall

2) *Numerical Recipes: Art of Scientific Computing* by William Press, Saul A. Teukolsky , William T. Vetterling and Brian P. Flannery, Cambridge University Press, (2007).

3)*Fourier transform & its applications* by Ronald Bracewell, Tata-McGraw-Hill, Third Edition

4)*Fast fourier transform and its applications* by Oran Brigham, Prentice-Hall.

Thank you

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